Suggested solution of HW3

Q6 (a) We prove it by induction. When $n = 1$, the assertion holds trivially. Suppose when $n = k$, $x_k \leq 3$ and $x_k > x_{k-1}$. Then

$$x_{k+1} = \frac{1}{2}(x_k + 3) > \frac{1}{2}(x_{k-1} + 3) = x_k.$$

And

$$x_{k+1} = \frac{1}{2}(x_k + 3) \leq 3.$$

By induction, we have for all $n \in \mathbb{N}$,

$$x_n < x_{n+1} \quad \text{and} \quad x_n \leq 3.$$

By monotone convergence theorem, $x_n \to \alpha$ for some $\alpha \in \mathbb{R}$. In particular, by taking limit both side, we have

$$\alpha = \frac{1}{2}(\alpha + 3)$$

meaning that $\alpha = 3$.

(b)

$$|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|.$$

Hence, for all $k, n \in \mathbb{N}$,

$$\begin{aligned} |x_{m+n} - x_n| &\leq \sum_{k=1}^{m} |x_{n+k} - x_{n+k-1}| \\ &= 2^{1-n}|x_2 - x_1| \cdot \sum_{k=1}^{m} 2^{-k}. \end{aligned}$$

Hence, $\{x_k\}$ is a cauchy sequence. Therefore, is convergent. Argue in a similar to above, we may conclude that $x_n \to 3$.

Q7 When $n = 1$, we can find $a_1 \in A$ such that $s - 1 < a_1$. When $n = 2$, there exists $\tilde{a}_2 \in A$ such that $s - \frac{1}{2} < \tilde{a}_2$. Repeat the step inductively, we can find a_n such that for all n,

$$s - \frac{1}{n} < a_n.$$

Denote $b_n = \sup\{a_1, a_2, ..., a_n\}$. Then $\{b_n\}$ is the desired sequence which converges to s.

Q8 As the sets are in descending order, the monotonicity is clear. Suppose

$$\limsup_n x_n = \liminf_n x_n = x.$$

Let $\epsilon > 0$, there is N such that for all $n > N$,

$$x \leq \sup_{k>n} x_k \leq x + \epsilon, \quad \text{and} \quad x - \epsilon \leq \inf_{k>n} x_k \leq x.$$

In particular, for all $n > N$,

$$x - \epsilon \leq x_n \leq x + \epsilon.$$